

A second variation formula for Perelman's \mathcal{W} -functional along the modified Kähler-Ricci flow

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January 5, 2012

Abstract

We show a quite simple second variation formula for Perelman's \mathcal{W} -functional along the modified Kähler-Ricci flow over Fano manifolds.

1 Introduction

In a celebrated preprint Grigory Perelman [Per] introduced an entropy type functional denoted by \mathcal{W} which gradient flow coincides with Perelman's modified Ricci flow. In the Fano case this flow coincides with Perelman's modified Kähler-Ricci flow. The total second variation of this functional plays an important role in the study of the stability and convergence of the Kähler-Ricci flow over Fano manifolds. Total second variation formulas at a shrinking Ricci soliton point were obtained independently by Cao-Hamilton-Ilmanen [C-H-I], Cao-Zhu [Ca-Zhu] and Tian-Zhu [Ti-Zhu]. The work of Cao-Zhu [Ca-Zhu] is based on the previous work of Cao-Hamilton-Ilmanen [C-H-I]. Important applications to the stability and the convergence of the Kähler-Ricci flow over Fano manifolds were given by Tian-Zhu [Ti-Zhu] and by Tian-Zhang-Zhang-Zhu [T-Z-Z-Z]. Total second variation formulas at arbitrary points in the space of metrics can be found in [Pal]. The formulas in [Pal] are of different nature from the ones obtained by [C-H-I], [Ca-Zhu] and [Ti-Zhu].

In this paper we show a very simple second variation formula for Perelman's \mathcal{W} -functional along the modified Kähler-Ricci flow over Fano manifolds.

Our computation concern the Kähler-Ricci flow but we feel that the second variation formula is more meaningful for the modified Kähler-Ricci flow. In any case one can switch easily from one formula relative to a flow to the other thanks to the invariance by diffeomorphisms of the functional \mathcal{W} . It is some how surprising that we could not prove a simple second variation formula by working directly with the modified Kähler-Ricci flow.

The key computation which allows us to obtain our second variation formula is based on a very surprising integration by parts formula along the Kähler-Ricci flow. It turns out that the corresponding version of this formula along the modified Kähler-Ricci flow can be obtained using Perelman's version of the twice contracted Bianchi identity. The details about this last fact are explained in section 3.

We expect convexity of the functional \mathcal{W} along the Kähler-Ricci flow. Our formula should be considered as a first step in this direction. It should be clear for the experts that convexity along the flow implies quite natural and strong convergence results. We explain in detail below the set-up and our result.

In this paper we will adopt the sign convention $\Delta := \operatorname{div} \nabla$ in order to not generate confusion in some standard formulas along the Kähler-Ricci flow.

Let (X, g) be a compact oriented (for simplicity) Riemannian manifold of dimension $2n$ and f a smooth real valued function over X . We remind that Perelman's \mathcal{W} -functional [Per] is defined (up to a constant) by the formula

$$\mathcal{W}(g, f) := \int_X \left(|\nabla_g f|_g^2 + \operatorname{Scal}_g + 2f - 2n \right) e^{-f} dV_g.$$

It rewrites also as

$$\begin{aligned} \mathcal{W}(g, f) &= \int_X \left(\Delta_g f + \operatorname{Scal}_g + 2f - 2n \right) e^{-f} dV_g \\ &= \int_X 2H(g, f) e^{-f} dV_g. \end{aligned}$$

where

$$2H(g, f) := 2\Delta_g f - |\nabla_g f|_g^2 + \operatorname{Scal}_g + 2f - 2n.$$

(We use here the standard identity $\Delta_g e^{-f} = (|\nabla_g f|_g^2 - \Delta_g f) e^{-f}$.) The importance of the therm H is known from Perelman's work [Per] (see the subsection 4.1 in the appendix) and it will appear also in our second variation formula.

Let (X, J_0) be a Fano manifold and let $(\hat{g}_t)_{t \geq 0}$ be the Kähler-Ricci flow

$$\frac{d}{dt} \hat{g}_t = -\operatorname{Ric}(\hat{g}_t) + \hat{g}_t,$$

with associated symplectic form $\hat{\omega}_t := \hat{g}_t J_0 \in 2\pi c_1(X)$. Consider also a solution $(\hat{f}_t)_{t \in [0, T]}$ of Perelman's backward heat equation

$$2 \frac{d}{dt} \hat{f}_t = -\Delta_{\hat{g}_t} \hat{f}_t + |\nabla_{\hat{g}_t} \hat{f}_t|_{\hat{g}_t}^2 - \operatorname{Scal}_{\hat{g}_t} + 2n, \quad (1)$$

and set $\Omega := e^{-\hat{f}_0} dV_{\hat{g}_0}$. Consider now the flow of diffeomorphisms $(\Psi_t)_{t \in [0, T]}$ given by the equation

$$2 \frac{d}{dt} \Psi_t = - \left(\nabla_{\hat{g}_t} \hat{f}_t \right) \circ \Psi_t, \quad (2)$$

and set $g_t := \Psi_t^* \hat{g}_t, J_t := \Psi_t^* J_0, \omega_t := \Psi_t^* \hat{\omega}_t, f_t := \hat{f}_t \circ \Psi_t$. Then hold the evolution formulas

$$\begin{aligned}\dot{g}_t &:= \frac{d}{dt} g_t = -\text{Ric}(g_t) - \nabla_{g_t} d f_t + g_t, \\ \dot{\omega}_t &:= \frac{d}{dt} \omega_t = -\text{Ric}_{J_t}(\omega_t) - i \partial_{J_t} \bar{\partial}_{J_t} f_t + \omega_t,\end{aligned}$$

and

$$2 \frac{d}{dt} f_t = -\Delta_{g_t} f_t - \text{Scal}(g_t) + 2n.$$

This last rewrites as

$$2 \frac{d}{dt} f_t = \text{Tr}_{\omega_t} \frac{d}{dt} \omega_t,$$

which is equivalent to the volume form preserving condition $e^{-f_t} dV_{g_t} = \Omega$. The flow of Kähler structures $(X, J_t, \omega_t)_{t \in [0, T]}$ is called Perelman's modified Kähler-Ricci flow. We set $\mathcal{W}_t := \mathcal{W}(g_t, f_t)$ along the modified Kähler-Ricci flow and we remind Perelman's [Per] fundamental identity

$$\dot{\mathcal{W}}_t = \int_X |\dot{g}_t|_t^2 \Omega. \quad (3)$$

Indeed this is equivalent to Perelman's monotony [Per] of the \mathcal{W} -functional along the Kähler-Ricci flow thanks to the invariance of the \mathcal{W} -functional under the action of diffeomorphisms. In our case it gives $\mathcal{W}_t = \mathcal{W}(\hat{g}_t, \hat{f}_t)$. With this notations our result states as follows.

Theorem 1. *Along the modified Kähler-Ricci flow $(X, J_t, \omega_t)_{t \in [0, T]}$ hold the second variation formula*

$$\ddot{\mathcal{W}}_t = \int_X \left[2 \langle \dot{\omega}_t \cdot \text{Rm}_{g_t}, \dot{\omega}_t \rangle_t + 2 |\nabla_t H_t|_t^2 - |\nabla_t \dot{g}_t|_t^2 \right] \Omega,$$

where $\text{Rm}_{g_t} \in C^\infty(X, \text{End}(\Lambda^2 T_X))$ denotes the Riemann curvature operator and $H_t := H(g_t, f_t)$.

2 The second variation of \mathcal{W} along the Kähler-Ricci flow

All the computations in this section are done with respect to the Kähler-Ricci flow (X, J_0, \hat{g}_t) . From now on all the complex operators will depend on the complex structure J_0 . Therefore we will suppress the dependence on J_0 when it is redundant. Moreover all the covariant derivatives and norms are computed with respect to the Kähler-Ricci flow. Thus for notation simplicity we will drop

the dependence on the evolving metric and on the time. Along the Kähler-Ricci flow we define the heat operator

$$\square := \Delta - 2 \frac{\partial}{\partial t}$$

and the conjugate heat operator

$$\square^* := \Delta + 2 \frac{\partial}{\partial t} - \text{Scal} + 2n.$$

The terminology is justified by the formula

$$2 \frac{d}{dt} \int_X (a b) dV_{\hat{g}} = - \int_X (\square a b - a \square^* b) dV_{\hat{g}},$$

for any $a, b \in C^\infty(X \times \mathbb{R}_{\geq 0}, \mathbb{R})$. We observe that with this notations the backward heat equation (1) is equivalent to the equation

$$\square^* e^{-\hat{f}} = 0. \quad (4)$$

For any $\alpha \in \Lambda_{\mathbb{R}}^2 T_X^*$ we define the endomorphism $\alpha^* := \hat{\omega}^{-1} \alpha$. Let now for notation simplicity

$$\text{Ric} := \text{Ric}_{J_0}(\hat{\omega}_t),$$

$$\underline{\text{Ric}} := \text{Ric} + i\partial\bar{\partial}\hat{f},$$

$$\alpha := \hat{\omega} - \underline{\text{Ric}},$$

$$A := \bar{\partial} \nabla \hat{f}.$$

Thus time deriving Perelman's monotony formula for the \mathcal{W} -functional along the Kähler-Ricci flow

$$\dot{\mathcal{W}} = \int_X (|\alpha^*|^2 + |A|^2) e^{-\hat{f}} dV_{\hat{g}},$$

(see formula (20) in the subsection 4.1 of the appendix) we obtain

$$\ddot{\mathcal{W}} = - \frac{1}{2} \int_X \square (|\alpha^*|^2 + |A|^2) e^{-\hat{f}} dV_{\hat{g}}. \quad (5)$$

In the next subsections we will expand the integrand therm.

2.1 Computation of the time variation of the squared norms

We introduce first a notation. Let $E := \text{End}_{\mathbb{C}}(T_{X, J_0})$. For all $\xi \in T_X$ we define the J_0 -anti-linear operator $\bar{\partial}_E \text{Ric}^* \cdot \xi$ as

$$(\bar{\partial}_E \text{Ric}^* \cdot \xi)(\eta) := (\bar{\partial}_{E, \eta} \text{Ric}^*)(\xi), \quad \forall \eta \in T_X.$$

The evolution equation of the Ricci form Ric and the evolution equation (1) of \hat{f} imply the identity

$$2 \frac{d}{dt} \underline{\text{Ric}} = i\partial\bar{\partial} \left(\text{Scal} + 2 \frac{d}{dt} \hat{f} \right) = i\partial\bar{\partial} \left(|\nabla \hat{f}|^2 - \Delta \hat{f} \right).$$

Moreover

$$\begin{aligned} \frac{d}{dt} \underline{\text{Ric}} &= \left(\frac{d}{dt} \hat{\omega} \right) \underline{\text{Ric}}^* + \hat{\omega} \frac{d}{dt} \underline{\text{Ric}}^* \\ &= \underline{\text{Ric}} - \text{Ric} \underline{\text{Ric}}^* + \hat{\omega} \frac{d}{dt} \underline{\text{Ric}}^*, \end{aligned}$$

and thus

$$\frac{d}{dt} \underline{\text{Ric}}^* = \left(\frac{d}{dt} \underline{\text{Ric}} \right)^* + \text{Ric}^* \underline{\text{Ric}}^* - \underline{\text{Ric}}^*.$$

The fact that the endomorphism $\alpha^* = \mathbb{I} - \underline{\text{Ric}}^*$ is \hat{g} -symmetric implies the identity $|\alpha^*|^2 = \text{Tr}_{\mathbb{R}}(\alpha^*)^2$. Time deriving this identity we infer

$$\begin{aligned} \frac{d}{dt} |\alpha^*|^2 &= - \text{Tr}_{\mathbb{R}} \left(\frac{d}{dt} \underline{\text{Ric}}^* \alpha^* + \alpha^* \frac{d}{dt} \underline{\text{Ric}}^* \right) \\ &= - 2 \text{Tr}_{\mathbb{R}} \left(\alpha^* \frac{d}{dt} \underline{\text{Ric}}^* \right) \\ &= - 2 \left\langle \alpha^*, \left(\frac{d}{dt} \underline{\text{Ric}} \right)^* + \text{Ric}^* \underline{\text{Ric}}^* - \underline{\text{Ric}}^* \right\rangle. \end{aligned}$$

In order to obtain the third equality we need to observe the elementary identities

$$\begin{aligned} \langle \text{Ric}^* \underline{\text{Ric}}^*, \alpha^* \rangle &= \text{Tr}_{\mathbb{R}} (\text{Ric}^* \underline{\text{Ric}}^* \alpha^*) \\ &= \text{Tr}_{\mathbb{R}} (\alpha^* \text{Ric}^* \underline{\text{Ric}}^*) \\ &= \langle \alpha^*, \underline{\text{Ric}}^* \text{Ric}^* \rangle. \end{aligned} \tag{6}$$

In conclusion hold the identity

$$\frac{d}{dt} |\alpha^*|^2 = \left\langle \alpha^*, \left[i\partial\bar{\partial}(\Delta \hat{f} - |\nabla \hat{f}|^2) \right]^* - 2 \text{Ric}^* \underline{\text{Ric}}^* + 2 \underline{\text{Ric}} \right\rangle. \tag{7}$$

We compute now the time variation of the operator A . Time deriving the definition of the gradient of \hat{f} we infer

$$\begin{aligned} d \left(\frac{d}{dt} \hat{f} \right) &= \left(\frac{d}{dt} \nabla \hat{f} \right) \lrcorner \hat{g} + \nabla \hat{f}_t \lrcorner \frac{d}{dt} \hat{g} \\ &= \left(\frac{d}{dt} \nabla \hat{f} \right) \lrcorner \hat{g} + d\hat{f} - \nabla \hat{f} \lrcorner \text{Ric}(\hat{g}), \end{aligned}$$

and thus using the elementary identity $\text{Ric}(\hat{g}) = -\text{Ric} \cdot J_0$ we obtain

$$\frac{d}{dt} \nabla \hat{f} = \nabla \left(\frac{d}{dt} \hat{f} - \hat{f} \right) + \text{Ric}^* \cdot \nabla \hat{f}.$$

We deduce the expression

$$\dot{A} := \frac{d}{dt} A = \bar{\partial} \nabla \left(\frac{d}{dt} \hat{f} - \hat{f} \right) + \bar{\partial}_E \text{Ric}^* \cdot \nabla \hat{f} + \text{Ric}^* A.$$

On the other hand $|A|^2 = \text{Tr}_{\mathbb{R}} A^2$ since A is also \hat{g} -symmetric. Time deriving this identity we infer the equalities

$$\begin{aligned} \frac{d}{dt} |A|^2 &= \text{Tr}_{\mathbb{R}} (\dot{A} A + A \dot{A}) \\ &= 2 \left\langle A, \bar{\partial} \nabla \left(\frac{d}{dt} \hat{f} - \hat{f} \right) + \bar{\partial}_E \text{Ric}^* \cdot \nabla \hat{f} \right\rangle \\ &\quad + 2 \langle A^2, \text{Ric}^* \rangle. \end{aligned}$$

In order to obtain the last term we use the identity

$$\text{Tr}_{\mathbb{R}} (A \text{Ric}^* A) = \langle A^2, \text{Ric}^* \rangle. \quad (8)$$

We infer from the evolution equation (1) of \hat{f} the formula

$$\begin{aligned} \frac{d}{dt} |A|^2 &= \left\langle A, \bar{\partial} \nabla \left(|\nabla \hat{f}|^2 - \Delta \hat{f} - \text{Scal} - 2 \hat{f} \right) \right\rangle \\ &\quad + 2 \left\langle A, \bar{\partial}_E \text{Ric}^* \cdot \nabla \hat{f} \right\rangle + 2 \langle A^2, \text{Ric}^* \rangle. \end{aligned} \quad (9)$$

2.2 Computation of the Laplacian of the squared norms

At an arbitrary space time point $(x_0, t_0) \in X \times [0, T]$ we fix J_0 -holomorphic and \hat{g}_{t_0} -geodesic coordinates (z_1, \dots, z_n) centered at the point x_0 and we set $\zeta_k := \frac{\partial}{\partial z_k}$. At the time t_0 let $\nabla_r := \nabla_{\zeta_r}$, $\nabla_{\bar{r}} := \nabla_{\bar{\zeta}_r}$ and set $e_r := \zeta_r + \bar{\zeta}_r$. Expanding and canceling we obtain the expression

$$\Delta = \nabla_{e_r} \nabla_{e_r} + \nabla_{J_0 e_r} \nabla_{J_0 e_r} = 2 \nabla_r \nabla_{\bar{r}} + 2 \nabla_{\bar{r}} \nabla_r. \quad (10)$$

Part I. Let

$$R_{k, \bar{l}} := \text{Ric}(\zeta_k, \bar{\zeta}_l),$$

$$R'_{k, \bar{l}} := \underline{\text{Ric}}(\zeta_k, \bar{\zeta}_l).$$

With this notations hold the local expression

$$\underline{\text{Ric}}^* = \mathcal{R}_{k, \bar{l}} \zeta_k^* \otimes \bar{\zeta}_l + \overline{\mathcal{R}_{k, \bar{l}}} \bar{\zeta}_k^* \otimes \bar{\zeta}_l, \quad \mathcal{R}_{k, \bar{l}} = 2 R'_{k, \bar{p}} \hat{\omega}^{p, \bar{l}},$$

where

$$\hat{\omega} = \frac{i}{2} \omega_{k,\bar{l}} \zeta_k^* \wedge \bar{\zeta}_l^* .$$

Using the expression (10), the vanishing properties of the complexified Levi-Civita connection and the previous identities we infer the expressions at the space time point (x_0, t_0)

$$\begin{aligned} \Delta \underline{\text{Ric}}^* &= 4 \partial_{r,\bar{r}}^2 \mathcal{R}_{k,\bar{l}} \zeta_k^* \otimes \zeta_l + 4 \partial_{r,\bar{r}}^2 \overline{\mathcal{R}_{k,\bar{l}}} \bar{\zeta}_k^* \otimes \bar{\zeta}_l \\ &+ 2 \mathcal{R}_{k,\bar{l}} \nabla_{\bar{r}} \nabla_r \zeta_k^* \otimes \zeta_l + 2 \mathcal{R}_{k,\bar{l}} \zeta_k^* \otimes \nabla_{\bar{r}} \nabla_r \zeta_l \\ &+ 2 \mathcal{R}_{l,\bar{k}} \nabla_r \nabla_{\bar{r}} \bar{\zeta}_k^* \otimes \bar{\zeta}_l + 2 \mathcal{R}_{l,\bar{k}} \bar{\zeta}_k^* \otimes \nabla_r \nabla_{\bar{r}} \bar{\zeta}_l \\ &= 4 \partial_{r,\bar{r}}^2 \mathcal{R}_{k,\bar{l}} \zeta_k^* \otimes \zeta_l + 4 \partial_{r,\bar{r}}^2 \overline{\mathcal{R}_{k,\bar{l}}} \bar{\zeta}_k^* \otimes \bar{\zeta}_l \\ &+ 2 (R_{k,\bar{p}} \mathcal{R}_{p,\bar{l}} - \mathcal{R}_{k,\bar{p}} R_{p,\bar{l}}) \zeta_k^* \otimes \zeta_l \\ &+ 2 (\mathcal{R}_{l,\bar{p}} R_{p,\bar{k}} - R_{l,\bar{p}} \mathcal{R}_{p,\bar{k}}) \bar{\zeta}_k^* \otimes \bar{\zeta}_l \\ &= 4 \partial_{r,\bar{r}}^2 \mathcal{R}_{k,\bar{l}} \zeta_k^* \otimes \zeta_l + 4 \partial_{r,\bar{r}}^2 \overline{\mathcal{R}_{k,\bar{l}}} \bar{\zeta}_k^* \otimes \bar{\zeta}_l \\ &+ \underline{\text{Ric}}^* \text{Ric}^* - \text{Ric}^* \underline{\text{Ric}}^* . \end{aligned}$$

At this point we need the following elementary fact.

Lemma 1. *At the space time point (x_0, t_0) hold the identity*

$$\partial_{r,\bar{r}}^2 \mathcal{R}_{k,\bar{l}} = \partial_{k,\bar{l}}^2 \mathcal{R}_{r,\bar{r}} - \mathcal{R}_{r,\bar{p}} \text{Rm}_{k,\bar{l}}^{r,\bar{p}} + 2 \mathcal{R}_{k,\bar{p}} R_{p,\bar{l}} ,$$

where

$$\text{Rm} = \text{Rm}_{k,\bar{l}}^{r,\bar{p}} (\zeta_k^* \wedge \bar{\zeta}_l^*) \otimes (\zeta_r \wedge \bar{\zeta}_p) , \quad \text{Rm}_{k,\bar{l}}^{r,\bar{p}} = -2 \partial_{k,\bar{l}}^2 \hat{\omega}_{p,\bar{r}} .$$

Proof. Space deriving twice the expression of the coefficients $R_{k,\bar{l}}$ and using the Kähler symmetry relations we find at the space time point (x_0, t_0) the expression

$$4 \partial_{j,\bar{k}}^2 R_{h,\bar{s}} = -4 \partial_{j,\bar{k},h,\bar{s}}^4 \hat{\omega}_{l,\bar{l}} + \text{Rm}_{l,\bar{r}}^{s,\bar{h}} \text{Rm}_{j,\bar{k}}^{l,\bar{r}} + \text{Rm}_{l,\bar{r}}^{k,\bar{h}} \text{Rm}_{j,\bar{s}}^{l,\bar{r}} ,$$

which implies the symmetry of the indices k, s for $\partial_{j,\bar{k}}^2 R_{h,\bar{s}}$. The symmetry of the indices j, h follows by conjugation. Moreover at the space time point (x_0, t_0) hold

$$\begin{aligned} \partial_{r,\bar{s}}^2 \mathcal{R}_{k,\bar{l}} &= 2 \partial_{r,\bar{s}}^2 R'_{k,\bar{l}} - 2 R'_{k,\bar{p}} \partial_{r,\bar{s}}^2 \hat{\omega}_{p,\bar{l}} \\ &= 2 \partial_{r,\bar{s}}^2 R_{k,\bar{l}} + 2 \partial_{r,\bar{s},k,\bar{l}}^4 \hat{f} - 2 R'_{k,\bar{p}} \partial_{r,\bar{s}}^2 \hat{\omega}_{p,\bar{l}} . \end{aligned}$$

We infer

$$\partial_{r,\bar{s}}^2 \mathcal{R}_{k,\bar{l}} = \partial_{k,\bar{l}}^2 \mathcal{R}_{r,\bar{s}} + 2 \mathcal{R}_{r,\bar{p}} \partial_{k,\bar{l}}^2 \hat{\omega}_{p,\bar{s}} - 2 \mathcal{R}_{k,\bar{p}} \partial_{r,\bar{s}}^2 \hat{\omega}_{p,\bar{l}},$$

and thus the required conclusion. \square

From lemma 1 we deduce the identity along the Kähler-Ricci flow

$$\Delta \underline{\text{Ric}}^* = (i\partial\bar{\partial} \text{Tr}_{\mathbb{R}} \underline{\text{Ric}}^* - 2 \underline{\text{Ric}} \text{Rm})^* + \underline{\text{Ric}}^* \text{Ric}^* + \text{Ric}^* \underline{\text{Ric}}^*. \quad (11)$$

Part II. We remind first the local expression

$$A = A_{k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + \overline{A_{k,\bar{l}}} \zeta_l^* \otimes \bar{\zeta}_k,$$

$$A_{k,\bar{l}} = 2 \hat{\omega}^{p,\bar{k}} \left[\partial_{l,\bar{p}}^2 \hat{f} - \partial_{\bar{l}} \hat{\omega}_{j,\bar{p}} \hat{\omega}^{r,\bar{j}} \partial_{\bar{r}} \hat{f} \right].$$

(See the subsection 4.2 in the appendix.) As in part I we infer the expressions at the space time point (x_0, t_0)

$$\begin{aligned} \Delta A &= 4 \partial_{r,\bar{r}}^2 A_{k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + 4 \partial_{r,\bar{r}}^2 \overline{A_{k,\bar{l}}} \zeta_l^* \otimes \bar{\zeta}_k \\ &+ 2 A_{k,\bar{l}} \nabla_r \nabla_{\bar{r}} \bar{\zeta}_l^* \otimes \zeta_k + 2 \overline{A_{k,\bar{l}}} \zeta_l^* \otimes \nabla_r \nabla_{\bar{r}} \bar{\zeta}_k \\ &+ 2 A_{k,\bar{l}} \bar{\zeta}_l^* \otimes \nabla_{\bar{r}} \nabla_r \zeta_k + 2 \overline{A_{k,\bar{l}}} \nabla_{\bar{r}} \nabla_r \zeta_l^* \otimes \bar{\zeta}_k \\ &= 4 \partial_{r,\bar{r}}^2 A_{k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + 4 \partial_{r,\bar{r}}^2 \overline{A_{k,\bar{l}}} \zeta_l^* \otimes \bar{\zeta}_k \\ &+ 2 (A_{k,\bar{p}} R_{p,\bar{l}} - A_{p,\bar{l}} R_{p,\bar{k}}) \bar{\zeta}_l^* \otimes \zeta_k \\ &+ 2 (\overline{A_{k,\bar{p}}} R_{l,\bar{p}} - \overline{A_{p,\bar{l}}} R_{k,\bar{p}}) \zeta_l^* \otimes \bar{\zeta}_k. \end{aligned}$$

Expanding the second order derivative we infer for all indices k, l the expression

$$\partial_{r,\bar{r}}^2 A_{k,\bar{l}} = 2 \partial_{r,\bar{r},\bar{k},\bar{l}}^4 \hat{f} + 2 \partial_{\bar{l}} R_{p,\bar{k}} \partial_{\bar{p}} \hat{f} + 2 R_{p,\bar{k}} \partial_{l,\bar{p}}^2 \hat{f}.$$

By plunging this in the previous expression and canceling the adequate terms we obtain

$$\begin{aligned} \Delta A &= \bar{\partial} \nabla \Delta \hat{f} + 2 \bar{\partial}_E \text{Ric}^* \cdot \nabla \hat{f} \\ &+ 4 \partial_{l,\bar{p}}^2 \hat{f} R_{p,\bar{k}} \bar{\zeta}_l^* \otimes \zeta_k + 4 R_{k,\bar{p}} \partial_{p,\bar{l}}^2 \hat{f} \zeta_l^* \otimes \bar{\zeta}_k \\ &+ 4 \partial_{k,\bar{p}}^2 \hat{f} R_{p,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + 4 R_{l,\bar{p}} \partial_{p,\bar{k}}^2 \hat{f} \zeta_l^* \otimes \bar{\zeta}_k, \end{aligned}$$

and thus the identity

$$\Delta A = \bar{\partial} \nabla \Delta \hat{f} + 2 \bar{\partial}_E \text{Ric}^* \cdot \nabla \hat{f} + \text{Ric}^* A + A \text{Ric}^* . \quad (12)$$

In conclusion the identities (11), (12), (6) (at the level of scalar products) and (8) imply

$$\begin{aligned} \Delta (|\alpha^*|^2 + |A|^2) &= 2|\nabla \alpha^*|^2 + 2|\nabla A|^2 \\ &- 2 \langle \alpha^*, \Delta \underline{\text{Ric}}^* \rangle + 2 \langle A, \Delta A \rangle \\ &= 2|\nabla \alpha^*|^2 + 2|\nabla A|^2 \\ &- 2 \left\langle \alpha^*, \left[i \partial \bar{\partial} (\text{Scal} + \Delta \hat{f}) \right]^* \right\rangle \\ &+ 4 \langle \alpha^*, (\underline{\text{Ric}} \text{Rm})^* - \text{Ric}^* \underline{\text{Ric}}^* \rangle \\ &+ 2 \left\langle A, \bar{\partial} \nabla \Delta \hat{f} + 2 \bar{\partial}_E \text{Ric}^* \cdot \nabla \hat{f} \right\rangle \\ &+ 4 \langle A^2, \text{Ric}^* \rangle . \end{aligned}$$

2.3 Evolution of the squared norms

The previous expression of the Laplacian of the squared norms combined with the formulas (7) and (9) provides the identity

$$\begin{aligned} \square (|\alpha^*|^2 + |A|^2) &= 2|\nabla \alpha^*|^2 + 2|\nabla A|^2 \\ &- 2 \left\langle \alpha^*, \left[i \partial \bar{\partial} (2 \Delta \hat{f} - |\nabla \hat{f}|^2 + \text{Scal}) \right]^* \right\rangle \\ &+ 2 \left\langle A, \bar{\partial} \nabla (2 \Delta \hat{f} - |\nabla \hat{f}|^2 + \text{Scal} + 2 \hat{f}) \right\rangle \\ &+ 4 \langle \alpha^*, (\underline{\text{Ric}}^* \text{Rm})^* \rangle - 4 \langle \alpha^*, \underline{\text{Ric}}^* \rangle . \end{aligned}$$

Thus if we set

$$2 \hat{H} := 2 H(\hat{g}, \hat{f}) = \square \hat{f} + 2 \hat{f} ,$$

we infer the evolution formula

$$\begin{aligned} \square (|\alpha^*|^2 + |A|^2) &= 2|\nabla \alpha^*|^2 + 2|\nabla A|^2 \\ &- 4 \left\langle \alpha^*, (i \partial \bar{\partial} \hat{H})^* \right\rangle + 4 \left\langle A, \bar{\partial} \nabla \hat{H} \right\rangle \\ &- 4 \langle \alpha \text{Rm}, \alpha \rangle . \end{aligned} \quad (13)$$

2.4 Integration by parts along the Kähler-Ricci flow

This last step is the key part of the proof of the second variation of Perelman's \mathcal{W} -functional along the Kähler-Ricci flow. Deriving the identity

$$0 = \int_X \left(\Delta \hat{H} - \nabla \hat{H} \cdot \nabla \hat{f} \right) e^{-\hat{f}} dV_{\hat{g}},$$

we obtain

$$0 = \int_X \square \left(\Delta \hat{H} - \nabla \hat{H} \cdot \nabla \hat{f} \right) e^{-\hat{f}} dV_{\hat{g}}, \quad (14)$$

thanks to (4). We expand now the integrand term in (14). We observe first the identity

$$\begin{aligned} \square \Delta \hat{H} &= \Delta \square \hat{H} + 2 \Delta \hat{H} - 2 \langle \text{Ric}, i\partial\bar{\partial} \hat{H} \rangle \\ &= 2 \Delta H_2 - 2 \langle \text{Ric}, i\partial\bar{\partial} \hat{H} \rangle, \end{aligned}$$

where

$$2 H_2 := \square \hat{H} + 2 \hat{H}.$$

(For more details on this type of computations see the proof of the formula (19) in the subsection 4.1 of the appendix). We compute next the heat term

$$\square (\nabla \hat{H} \cdot \nabla \hat{f}).$$

We observe first the expression of the time derivative

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \hat{H} \cdot \nabla \hat{f}) &= \frac{\partial}{\partial t} (d\hat{H} \cdot \nabla \hat{f}) \\ &= d \left(\frac{\partial}{\partial t} \hat{H} \right) \cdot \nabla \hat{f} + d\hat{H} \cdot \frac{\partial}{\partial t} \nabla \hat{f} \\ &= \nabla \left(\frac{\partial}{\partial t} \hat{H} \right) \cdot \nabla \hat{f} + \nabla \hat{H} \cdot \nabla \left(\frac{\partial}{\partial t} \hat{f} - \hat{f} \right) \\ &\quad + \text{Ric} (\nabla \hat{f}, J_0 \nabla \hat{H}). \end{aligned}$$

We expand the Laplacian

$$\begin{aligned} \Delta (\nabla \hat{H} \cdot \nabla \hat{f}) &= 8 \partial_{r,\bar{r}}^2 \left[\hat{\omega}^{k,\bar{l}} \left(\hat{H}_l \hat{f}_{\bar{k}} + \hat{f}_l \hat{H}_{\bar{k}} \right) \right] \\ &= -8 \partial_{r,\bar{r}}^2 \hat{\omega}_{k,\bar{l}} \left(\hat{H}_l \hat{f}_{\bar{k}} + \hat{f}_l \hat{H}_{\bar{k}} \right) \\ &\quad + 8 \partial_{r,\bar{r}}^2 \left(\hat{H}_l \hat{f}_{\bar{k}} + \hat{f}_l \hat{H}_{\bar{k}} \right) \end{aligned}$$

$$\begin{aligned}
&= 8 R_{k,\bar{l}} \left(\hat{H}_l \hat{f}_{\bar{k}} + \hat{f}_l \hat{H}_{\bar{k}} \right) \\
&+ 2 \partial_k \Delta \hat{H} \hat{f}_{\bar{k}} + 8 \hat{H}_{k,\bar{l}} \hat{f}_{\bar{k},l} \\
&+ 8 \hat{H}_{k,l} \hat{f}_{\bar{k},\bar{l}} + 2 \hat{H}_k \partial_{\bar{k}} \Delta \hat{f} \\
&+ 2 \partial_k \Delta \hat{f} \hat{H}_{\bar{k}} + 8 \hat{f}_{k,\bar{l}} \hat{H}_{\bar{k},l} \\
&+ 8 \hat{f}_{k,l} \hat{H}_{\bar{k},\bar{l}} + 2 \hat{f}_k \partial_{\bar{k}} \Delta \hat{H} \\
&= 2 \text{Ric}(\nabla \hat{H}, J_0 \nabla \hat{f}) + \nabla \Delta \hat{H} \cdot \nabla \hat{f} + \nabla \hat{H} \cdot \nabla \Delta \hat{f} \\
&+ 2 \langle i \partial \bar{\partial} \hat{H}, i \partial \bar{\partial} \hat{f} \rangle + 2 \langle \bar{\partial} \nabla \hat{H}, \bar{\partial} \nabla \hat{f} \rangle.
\end{aligned}$$

We infer the equality

$$\begin{aligned}
\Box \left(\nabla \hat{H} \cdot \nabla \hat{f} \right) &= \nabla \Box \hat{H} \cdot \nabla \hat{f} + \nabla \hat{H} \cdot \nabla \Box \hat{f} + 2 \nabla \hat{H} \cdot \nabla \hat{f} \\
&+ 2 \langle i \partial \bar{\partial} \hat{H}, i \partial \bar{\partial} \hat{f} \rangle + 2 \langle \bar{\partial} \nabla \hat{H}, \bar{\partial} \nabla \hat{f} \rangle \\
&= 2 \nabla H_2 \cdot \nabla \hat{f} + 2 |\nabla \hat{H}|^2 - 2 \nabla \hat{H} \cdot \nabla \hat{f} \\
&+ 2 \langle i \partial \bar{\partial} \hat{H}, i \partial \bar{\partial} \hat{f} \rangle + 2 \langle \bar{\partial} \nabla \hat{H}, \bar{\partial} \nabla \hat{f} \rangle.
\end{aligned}$$

Finally we obtain the identity

$$\begin{aligned}
\Box \left(\Delta \hat{H} - \nabla \hat{H} \cdot \nabla \hat{f} \right) &= 2 \Delta H_2 - 2 \nabla H_2 \cdot \nabla \hat{f} \\
&+ 2 \nabla \hat{H} \cdot \nabla \hat{f} - 2 |\nabla \hat{H}|^2 \\
&- 2 \langle \text{Ric} + i \partial \bar{\partial} \hat{f}, i \partial \bar{\partial} \hat{H} \rangle - 2 \langle A, \bar{\partial} \nabla \hat{H} \rangle.
\end{aligned}$$

By adding and subtracting the term $2 \Delta \hat{H}$ we obtain the evolution formula

$$\begin{aligned}
\Box \left(\Delta \hat{H} - \nabla \hat{H} \cdot \nabla \hat{f} \right) &= 2 \Delta H_2 - 2 \nabla H_2 \cdot \nabla \hat{f} \\
&+ 2 \nabla \hat{H} \cdot \nabla \hat{f} - 2 \Delta \hat{H} - 2 |\nabla \hat{H}|^2 \\
&+ 2 \langle \alpha^*, (i \partial \bar{\partial} \hat{H})^* \rangle - 2 \langle A, \bar{\partial} \nabla \hat{H} \rangle,
\end{aligned}$$

which combined with the identity (14) implies the formula

$$\int_X |\nabla \hat{H}|^2 e^{-\hat{f}} dV_{\hat{g}} = \int_X \left[\left\langle \alpha^*, (i\partial\bar{\partial}\hat{H})^* \right\rangle - \left\langle A, \bar{\partial} \nabla \hat{H} \right\rangle \right] e^{-\hat{f}} dV_{\hat{g}}, \quad (15)$$

via integration by parts. Combining the equality (15) with the identity (5) and with the evolution formula (13) we infer the second variation identity

$$\ddot{\mathcal{W}} = \int_X \left[2 \langle \alpha \text{Rm}, \alpha \rangle + 2 |\nabla \hat{H}|^2 - |\nabla \alpha^*|^2 - |\nabla A|^2 \right] e^{-\hat{f}} dV_{\hat{g}}.$$

By using the diffeomorphisms invariance of the integrals on the r.h.s. we infer the conclusion of theorem 1.

3 Interpretation of formula (15) along the modified Kähler-Ricci flow

We remind that the Ω -Bakry-Emery-Ricci tensor of a metric g is defined by the formula

$$\text{Ric}_g(\Omega) := \text{Ric}(g) + \nabla_g d \log \frac{dV_g}{\Omega}.$$

We denote by $\text{Ric}_g^*(\Omega) := g^{-1} \text{Ric}_g(\Omega)$ the endomorphism section associated to the Ω -Bakry-Emery-Ricci tensor. We set $f := \log \frac{dV_g}{\Omega}$ and we observe that the operator

$$\nabla_g^{*\Omega} := e^f \nabla_g^* (e^{-f} \bullet) = \nabla_g^* + \nabla_g f \lrcorner,$$

is the formal adjoint of ∇_g with respect to the scalar product $\int_X \langle \cdot, \cdot \rangle_g \Omega$.

We show now Perelman's version of the twice contracted differential Bianchi identity (see [Per])

$$\nabla_g^{*\Omega} \text{Ric}_g^*(\Omega) = \nabla_g (f - H), \quad (16)$$

In fact expanding the l.h.s we obtain

$$\begin{aligned} 2 \nabla_g^{*\Omega} \text{Ric}_g^*(\Omega) &= 2 \nabla_g^* \text{Ric}_g^*(\Omega) + 2 \text{Ric}_g^*(\Omega) \cdot \nabla_g f \\ &= 2 \nabla_g^* \text{Ric}_g^* + 2 \nabla_g^* \nabla_g^2 f \\ &\quad + 2 \text{Ric}_g^* \cdot \nabla_g f + 2 \nabla_g^2 f \cdot \nabla_g f \\ &= - \nabla_g \text{Scal}_g - 2 \Delta_g \nabla_g f \\ &\quad + 2 \text{Ric}_g^* \cdot \nabla_g f + \nabla_g |\nabla_g f|_g^2, \end{aligned}$$

thanks to the twice contracted Bianchi identity. Then formula (16) follows from the identity

$$\nabla_g \Delta_g f = \Delta_g \nabla_g f - \text{Ric}_g^* \cdot \nabla_g f .$$

We remind now that any smooth volume form $\Omega > 0$ over a complex manifold (X, J) of complex dimension n induces a hermitian metric h_Ω over the canonical bundle $K_{X,J} := \Lambda_J^{n,0} T_X^*$ given by the formula

$$h_\Omega(\alpha, \beta) := \frac{n! i^{n^2} \alpha \wedge \bar{\beta}}{\Omega} .$$

By abuse of notations we will denote by Ω^{-1} the metric h_Ω . The dual metric h_Ω^* on the anti canonical bundle $K_{X,J}^{-1} = \Lambda_J^{n,0} T_X$ is given by the formula

$$h_\Omega^*(\xi, \eta) = (-i)^{n^2} \Omega(\xi, \bar{\eta}) / n! .$$

Abusing notations again, we denote by Ω the dual metric h_Ω^* . We define the Ω -Ricci form

$$\text{Ric}_J(\Omega) := i \mathcal{C}_\Omega(K_{X,J}^{-1}) = -i \mathcal{C}_{\Omega^{-1}}(K_{X,J}) ,$$

where $\mathcal{C}_h(L)$ denotes the Chern curvature of a hermitian line bundle (L, h) . In particular $\text{Ric}_J(\omega) = \text{Ric}_J(\omega^n)$. We remind also that for any J -invariant Kähler metric g the associated symplectic form $\omega := gJ$ satisfies the elementary identity $\text{Ric}(g) = -\text{Ric}_J(\omega) J$. Moreover for all twice differentiable function u hold the identity

$$\nabla_g du = - (i \partial_J \bar{\partial}_J u) J + g \bar{\partial}_{T_{X,J}} \nabla_g u .$$

We infer the decomposition identity

$$\text{Ric}_g(\Omega) = -\text{Ric}_J(\Omega) J + g \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega} ,$$

and thus the identity

$$\text{Ric}_g^*(\Omega) = \text{Ric}_J^*(\Omega) + \bar{\partial}_{T_{X,J}} \nabla_g \log \frac{dV_g}{\Omega} , \quad (17)$$

where $\text{Ric}_J^*(\Omega) := \omega^{-1} \text{Ric}_J(\Omega)$. We will denote by $\langle \cdot, \cdot \rangle_\omega$ the hermitian product on T_X -valued forms induced by the hermitian metric on $T_{X,J}$. The formal adjoint of the $\partial_{T_{X,J}}^g$ -operator with respect to the L^2 -hermitian product $\int_X \langle \cdot, \cdot \rangle_\omega \Omega$, is the operator

$$\partial_{T_{X,J}}^{*,\Omega} := e^f \partial_{T_{X,J}}^{*g} (e^{-f} \bullet) .$$

In a similar way the formal adjoint of the $\bar{\partial}_{T_{X,J}}$ -operator with respect to the L^2 -hermitian product $\int_X \langle \cdot, \cdot \rangle_\omega \Omega$, is the operator

$$\bar{\partial}_{T_{X,J}}^{*,\Omega} := e^f \bar{\partial}_{T_{X,J}}^{*g} (e^{-f} \bullet) .$$

With this notations hold the decomposition formula

$$p \nabla_g^* \Omega = \partial_{T_{X,J}}^{*g,\Omega} + \bar{\partial}_{T_{X,J}}^{*g,\Omega},$$

at the level of T_X -valued p -forms. (See the appendix in [Pal]). We observe also that the identity $d \text{Ric}_J(\Omega) = 0$ is equivalent to the identity $\partial_J \text{Ric}_J(\Omega) = 0$, which in its turn is equivalent to the identity

$$\partial_{T_{X,J}}^g \text{Ric}_J^*(\Omega) = 0.$$

We deduce

$$\nabla_g^* \text{Ric}_{J_t}^*(\Omega) = \partial_{T_{X,J_t}}^{*g_t,\Omega} \text{Ric}_{J_t}^*(\Omega).$$

thanks to a basic Kähler identity. We observe now that by the diffeomorphisms invariance of the integrals formula (15) writes as

$$\begin{aligned} \int_X |\nabla_t H_t|_t^2 \Omega &= - \int_X \left\langle \text{Ric}_{J_t}^*(\Omega) - \mathbb{I}_{T_X}, \partial_{T_{X,J_t}}^{g_t} \nabla_t H_t \right\rangle_t \\ &\quad - \int_X \left\langle \bar{\partial}_{T_{X,J_t}} \nabla_t f_t, \bar{\partial}_{T_{X,J}} \nabla_t H_t \right\rangle_t \Omega, \end{aligned}$$

along the modified Kähler-Ricci flow. We show now this formula by using the previous considerations. Using the comparison of norms on $T_{X,J}$ -valued forms in the subsection 4.3 of the appendix we expand the integral term

$$\begin{aligned} &- \int_X \left[\left\langle \text{Ric}_{J_t}^*(\Omega), \partial_{T_{X,J_t}}^{g_t} \nabla_t H_t \right\rangle_t + \left\langle \bar{\partial}_{T_{X,J_t}} \nabla_t f_t, \bar{\partial}_{T_{X,J}} \nabla_t H_t \right\rangle_t \right] \Omega \\ &= - \frac{1}{2} \int_X \left[\left\langle \text{Ric}_{J_t}^*(\Omega), \partial_{T_{X,J_t}}^{g_t} \nabla_t H_t \right\rangle_{\omega_t} + \left\langle \partial_{T_{X,J_t}}^{g_t} \nabla_t H_t, \text{Ric}_{J_t}^*(\Omega) \right\rangle_{\omega_t} \right] \Omega \\ &\quad - \frac{1}{2} \int_X \left[\left\langle \bar{\partial}_{T_{X,J_t}} \nabla_t f_t, \bar{\partial}_{T_{X,J}} \nabla_t H_t \right\rangle_{\omega_t} + \left\langle \bar{\partial}_{T_{X,J_t}} \nabla_t H_t, \bar{\partial}_{T_{X,J}} \nabla_t f_t \right\rangle_{\omega_t} \right] \Omega \\ &= - \frac{1}{2} \int_X \left[\left\langle \partial_{T_{X,J_t}}^{*g_t,\Omega} \text{Ric}_{J_t}^*(\Omega), \nabla_t H_t \right\rangle_{\omega_t} + \left\langle \nabla_t H_t, \partial_{T_{X,J_t}}^{*g_t,\Omega} \text{Ric}_{J_t}^*(\Omega) \right\rangle_{\omega_t} \right] \Omega \\ &\quad - \frac{1}{2} \int_X \left[\left\langle \bar{\partial}_{T_{X,J_t}}^{*g_t,\Omega} \bar{\partial}_{T_{X,J_t}} \nabla_t f_t, \nabla_t H_t \right\rangle_{\omega_t} + \left\langle \nabla_t H_t, \bar{\partial}_{T_{X,J_t}}^{*g_t,\Omega} \bar{\partial}_{T_{X,J_t}} \nabla_t f_t \right\rangle_{\omega_t} \right] \Omega \\ &= - \int_X \left[\left\langle \partial_{T_{X,J_t}}^{*g_t,\Omega} \text{Ric}_{J_t}^*(\Omega), \nabla_t H_t \right\rangle_t + \left\langle \bar{\partial}_{T_{X,J_t}}^{*g_t,\Omega} \bar{\partial}_{T_{X,J_t}} \nabla_t f_t, \nabla_t H_t \right\rangle_t \right] \Omega \\ &= - \int_X \left[\left\langle \nabla_g^* \text{Ric}_{J_t}^*(\Omega), \nabla_t H_t \right\rangle_t + \left\langle \nabla_g^* \bar{\partial}_{T_{X,J_t}} \nabla_t f_t, \nabla_t H_t \right\rangle_t \right] \Omega \\ &= - \int_X \left\langle \nabla_g^* \text{Ric}_g^*(\Omega), \nabla_t H_t \right\rangle_t \Omega \\ &= \int_X \left[|\nabla_t H_t|_t^2 - \left\langle \nabla_t f_t, \nabla_t H_t \right\rangle_t \right] \Omega, \end{aligned}$$

thanks to basic Kähler identities and thanks to formulas (17) and (16). The required formula follows from the trivial identities.

$$\int_X \langle \nabla_t f_t, \nabla_t H_t \rangle_t \Omega = \int_X \Delta_t H_t \Omega = \int_X \left\langle \mathbb{I}_{T_X}, \partial_{T_X, J_t}^{g_t} \nabla_t H_t \right\rangle_t \Omega.$$

4 Appendix

4.1 The first variation of Perelman's \mathcal{W} functional along the Kähler-Ricci flow

Analogues of the following evolution formulas were obtained by Perelman [Per] in the Ricci flow case. For notation convenience we denote by $(g_t)_{t \geq 0}$ the Kähler-Ricci flow and with ω_t the corresponding symplectic forms.

Theorem 2. (Perelman) *Let X be a Fano manifold and let f be a solution of the conjugate heat equation*

$$2 \dot{f} = -\Delta f + |\nabla f|^2 + 2n - \text{Scal}, \quad (18)$$

along the Kähler-Ricci flow $(g_t)_{t \geq 0}$, over a time interval $[0, T]$. Then the function

$$2H := 2\Delta f - |\nabla f|^2 + \text{Scal} + 2f - 2n,$$

satisfies the evolution equation

$$2 \dot{H} = -\Delta H + 2 \nabla H \cdot \nabla f + |\text{Ric} + i\partial\bar{\partial}f - \omega_t|^2 + |\nabla^{1,0}\partial f|^2, \quad (19)$$

over the time interval $[0, T]$. Moreover on this interval hold the variation formula

$$\frac{d}{dt} \mathcal{W}(g_t, f_t) = \int_X \left[|\text{Ric} + i\partial\bar{\partial}f - \omega_t|^2 + |\nabla^{1,0}\partial f|^2 \right] e^{-f} dV_{g_t}. \quad (20)$$

Proof. We remind first that for any function $f \in C^\infty(X \times \mathbb{R}_{\geq 0}, \mathbb{R})$ hold the evolution formulas along the Kähler-Ricci flow

$$\frac{\partial}{\partial t} |\nabla f|^2 = -|\nabla f|^2 + \text{Ric}(\nabla f, J\nabla f) + 2 \nabla f \cdot \nabla f, \quad (21)$$

$$\begin{aligned} \Delta |\nabla f|^2 &= 2 \nabla \Delta f \cdot \nabla f + 2 |\nabla^{1,0}\partial f|^2 + 2 |\partial\bar{\partial}f|^2 \\ &+ 2 \text{Ric}(\nabla f, J\nabla f), \end{aligned} \quad (22)$$

$$\frac{\partial}{\partial t} \Delta f = -\Delta f + \langle \text{Ric}, i\partial\bar{\partial}f \rangle + \Delta \dot{f}. \quad (23)$$

We remind also that the scalar curvature evolves by the formula

$$2 \frac{\partial}{\partial t} \text{Scal} = \Delta \text{Scal} + 2 |\text{Ric}|^2 - 2 \text{Scal}. \quad (24)$$

Furthermore we observe the identity

$$2H = \square f + 2f. \quad (25)$$

By using the evolution equation (23) we get the equality

$$\begin{aligned} \square \Delta f &= \Delta^2 f + 2 \Delta f - 2 \langle \text{Ric}, i\partial\bar{\partial}f \rangle - 2 \Delta f \\ &= \Delta \square f + 2 \Delta f - 2 \langle \text{Ric}, i\partial\bar{\partial}f \rangle \\ &= 2 \Delta H - 2 \langle \text{Ric}, i\partial\bar{\partial}f \rangle, \end{aligned}$$

thanks to the identity (25). Moreover if we combine the evolution equations (21) and (22) we obtain

$$\begin{aligned} \square |\nabla f|^2 &= 2 \nabla \square f \cdot \nabla f + 2 |\nabla^{1,0} \partial f|^2 + 2 |\partial\bar{\partial}f|^2 + 2 |\nabla f|^2 \\ &= 4 \nabla H \cdot \nabla f + 2 |\nabla^{1,0} \partial f|^2 + 2 |\partial\bar{\partial}f|^2 - 2 |\nabla f|^2, \end{aligned}$$

thanks to the identity (25). We infer the expressions

$$\begin{aligned} 2 \square H &= 2 \square \Delta f - \square |\nabla f|^2 + \square \text{Scal} + 2 \square f \\ &= 4 \Delta H - 4 \langle \text{Ric}, i\partial\bar{\partial}f \rangle \\ &\quad - 4 \nabla H \cdot \nabla f - 2 |\nabla^{1,0} \partial f|^2 - 2 |\partial\bar{\partial}f|^2 + 2 |\nabla f|^2 \\ &\quad - 2 |\text{Ric}|^2 + 2 \text{Scal} + 4 H - 4 f \\ &= 4 \Delta H + 4 \Delta f + 4 \text{Scal} - 4 \langle \text{Ric}, i\partial\bar{\partial}f \rangle - 4 \nabla H \cdot \nabla f \\ &\quad - 2 |\nabla^{1,0} \partial f|^2 - 2 |\partial\bar{\partial}f|^2 - 2 |\text{Ric}|^2 - 4 n \\ &= 4 \Delta H - 4 \nabla H \cdot \nabla f \\ &\quad - 2 (B - 2 \Delta f + 2 n - 2 \text{Scal}). \end{aligned}$$

where

$$B := |\nabla^{1,0} \partial f|^2 + |\partial\bar{\partial}f|^2 + 2 \langle \text{Ric}, i\partial\bar{\partial}f \rangle + |\text{Ric}|^2.$$

Arranging the terms by means of the trivial identity $\text{Tr}_{\omega_t} \alpha = \langle \omega_t, \alpha \rangle$, with α a real $(1, 1)$ -form, we obtain the evolution equation

$$\begin{aligned} 2 \square H &= 4 \Delta H - 4 \nabla H \cdot \nabla f \\ &- 2 |\text{Ric} + i \partial \bar{\partial} f - \omega_t|^2 - 2 |\nabla^{1,0} \partial f|^2, \end{aligned}$$

which implies the evolution formula (19). We remind now that the evolution equation (18) rewrites as $\square^* e^{-f} = 0$. Thus time deriving the identity

$$\mathcal{W}(g_t, f_t) = \int_X 2 H e^{-f} dV_g,$$

we infer

$$\frac{d}{dt} \mathcal{W}(g_t, f_t) = - \int_X \square H e^{-f} dV_g,$$

which implies Perelman's variation formula (20). \square

4.2 Local expression of the complex anti-linear part of the Hessian

Let (X, J, ω) be a Kähler manifold and $u \in C^2(X, \mathbb{R})$. Let (z_1, \dots, z_n) be J -holomorphic coordinates and consider the local expression

$$\bar{\partial}_{T_{X,J}} \nabla_g u = A_{k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + \overline{A_{k,\bar{l}}} \zeta_l^* \otimes \bar{\zeta}_k,$$

where $\zeta_k := \frac{\partial}{\partial z_k}$. We want to find the expression of the coefficients $A_{k,\bar{l}}$ with respect to u . For this purpose we consider the identities

$$\nabla_g u = \nabla_{g,J}^{1,0} u + \nabla_{g,J}^{0,1} u$$

and

$$\nabla_{g,J}^{1,0} u \lrcorner \omega = i \bar{\partial}_J u.$$

If we write locally $\nabla_{g,J}^{1,0} u = \xi_k \zeta_k$ then the last identity writes locally as

$$\frac{i}{2} \omega_{l,\bar{k}} \xi_l \bar{\zeta}_k^* = i (\bar{\zeta}_k \cdot u) \bar{\zeta}_k^*.$$

We infer the expression $\xi_l = 2 \omega^{k,\bar{l}} \bar{\zeta}_k \cdot u$. Moreover by the definition of the operator $\bar{\partial}_{T_{X,J}}$ hold the identities

$$A_{k,\bar{l}} \zeta_k = \left[(\bar{\partial}_{T_{X,J}} \nabla_g u) \bar{\zeta}_l \right]_{J_t}^{1,0} = \left[\bar{\zeta}_l, \nabla_{g,J}^{1,0} u \right]_{J_t}^{1,0} = (\bar{\zeta}_l \cdot \xi_k) \zeta_k.$$

We infer the expressions

$$\begin{aligned} A_{k,\bar{l}} &= \bar{\zeta}_l \cdot \xi_k = 2 \bar{\zeta}_l \cdot \left(\omega^{r,\bar{k}} \bar{\zeta}_r \cdot f \right) \\ &= 2 \omega^{p,\bar{k}} \left[\bar{\zeta}_l \cdot \bar{\zeta}_p \cdot f_t - (\bar{\zeta}_l \cdot \omega_{j,\bar{p}}) \omega^{r,\bar{j}} \bar{\zeta}_r \cdot f_t \right]. \end{aligned}$$

4.3 Comparison of norms on $T_{X,J}$ -valued forms

Let (X, J, g) be a hermitian manifold. Let $\omega := gJ$ and let h^* be the corresponding hermitian metric over the complex vector bundle $T_{X,J}^*$. With respect to a local complex frame $(\zeta_k)_k \subset T_{X,J}^{1,0}$ we have the expression

$$h^* = 4 \sum_{k,l} \omega^{l\bar{k}} \zeta_k \otimes \bar{\zeta}_l .$$

We remind that if (V, J) is a complex vector space equipped with a hermitian metric h then the corresponding hermitian metric $h_{\mathbb{C}}$ over the complexified vector space $(V \otimes_{\mathbb{R}} \mathbb{C}, i)$ is defined by the formula

$$2h_{\mathbb{C}}(v, w) := h(v, \bar{w}) + \overline{h(\bar{v}, w)}, \quad v, w \in V \otimes_{\mathbb{R}} \mathbb{C} ,$$

where we still note by h the \mathbb{C} -linear extension of h . Thus $h_{\mathbb{C}}$ coincides with the sesquilinear extension over $V \otimes_{\mathbb{R}} \mathbb{C}$ of the Riemannian metric associated to h . We infer by the expression (31) in [Pal] of the Riemannian metric on the exterior products that the induced hermitian product on the vector bundle $\Lambda_J^{p,q} T_X^*$ is given by the formula

$$\begin{aligned} & \left\langle \Lambda_{j=1}^p \alpha_{1,j} \wedge \Lambda_{j=1}^q \beta_{1,j} , \Lambda_{j=1}^p \alpha_{2,j} \wedge \Lambda_{j=1}^q \beta_{2,j} \right\rangle \\ &= (p+q)! \det \left(2^{-1} h^*(\alpha_{1,j}, \bar{\alpha}_{2,l}) \right) \overline{\det \left(2^{-1} h^*(\beta_{1,j}, \beta_{2,l}) \right)} . \end{aligned}$$

Consider now an element

$$A \in T_{X,-J}^* \otimes_{\mathbb{C}} T_{X,J} \cong \Lambda_J^{0,1} T_X^* \otimes_{\mathbb{C}} T_{X,J} ,$$

and let $(e_k)_k \subset T_{X,J}$, $e_k := \zeta_k + \bar{\zeta}_k$ be the J -complex basis associated to $(\zeta_k)_k$. Then hold the local expression

$$A = A_{k,\bar{l}} \bar{\zeta}_l^* \otimes_J e_k = A_{k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k + \text{Conjugate} .$$

Assume from now on that the frame $(e_k)_k$ is h -orthonormal. On one side if one think of A as an element in $\text{End}_{\mathbb{R}}(T_X)$ then

$$|A|_g^2 = \text{Tr}_{\mathbb{R}}(A A_g^T) = 2 |A_{k,\bar{l}}|^2 ,$$

since

$$A_g^T = A_{l,\bar{k}} \bar{\zeta}_l^* \otimes \zeta_k + \text{Conjugate} .$$

On the other side

$$\begin{aligned} |A|_{\omega}^2 &\equiv |A|_{\Lambda_J^{0,1} T_X^* \otimes_{\mathbb{C}} T_{X,J}, \omega}^2 = \langle A_{k,\bar{l}} \bar{\zeta}_l^* , A_{k,\bar{p}} \bar{\zeta}_p^* \rangle \\ &= \frac{1}{2} \overline{h^*(\bar{A}_{k,\bar{l}} \bar{\zeta}_l^* , A_{k,\bar{p}} \bar{\zeta}_p^*)} \\ &= \frac{1}{2} |A_{k,\bar{l}}|^2 \cdot 4 , \end{aligned}$$

Thus $|A|_g^2 = |A|_\omega^2$. The same identity hold true for any

$$A \in T_{X,J}^* \otimes_{\mathbb{C}} T_{X,J} \cong \Lambda_J^{1,0} T_X^* \otimes_{\mathbb{C}} T_{X,J} .$$

In higher degrees there is a multiplicative factor involved. We consider for example $A \in \Lambda_J^{1,1} T_X^* \otimes_{\mathbb{C}} T_{X,J}$ and its local expression

$$A = i A_{p,k,\bar{l}} (\zeta_p^* \wedge \bar{\zeta}_l^*) \otimes \zeta_k + \text{Conjugate} .$$

Then

$$\begin{aligned} |A|_{\Lambda_J^{1,1} T_X^* \otimes_{\mathbb{C}} T_{X,J}, \omega}^2 &= \langle i A_{p,k,\bar{l}} \zeta_p^* \wedge \bar{\zeta}_l^*, i A_{r,k,\bar{h}} \zeta_r^* \wedge \bar{\zeta}_h^* \rangle \\ &= A_{p,k,\bar{l}} \bar{A}_{r,k,\bar{h}} \langle \zeta_p^* \wedge \bar{\zeta}_l^*, \zeta_r^* \wedge \bar{\zeta}_h^* \rangle \\ &= \frac{1}{2} |A_{p,k,\bar{l}}|^2 h^* (\zeta_p^*, \bar{\zeta}_p^*) \overline{h^* (\zeta_l^*, \bar{\zeta}_l^*)} \\ &= 8 |A_{p,k,\bar{l}}|^2 . \end{aligned}$$

On the other side if we think of A as an element of $\Lambda^2 T_X \otimes_{\mathbb{R}} T_X$ then

$$\begin{aligned} |A|_{\Lambda^2 T_X \otimes_{\mathbb{R}} T_X, g}^2 &= \text{Tr}_{\mathbb{R}} \left[(e_r \lrcorner A) (e_r \lrcorner A)_g^T \right] \\ &+ \text{Tr}_{\mathbb{R}} \left[(J e_r \lrcorner A) (J e_r \lrcorner A)_g^T \right] , \end{aligned}$$

and

$$\begin{aligned} e_r \lrcorner A &= i A_{r,k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k - i A_{l,k,\bar{r}} \zeta_l^* \otimes \zeta_k + \text{Conjugate} , \\ J e_r \lrcorner A &= - A_{r,k,\bar{l}} \bar{\zeta}_l^* \otimes \zeta_k - A_{l,k,\bar{r}} \zeta_l^* \otimes \zeta_k + \text{Conjugate} , \\ (e_r \lrcorner A)_g^T &= i A_{r,l,\bar{k}} \bar{\zeta}_l^* \otimes \zeta_k + i \bar{A}_{k,l,\bar{r}} \zeta_l^* \otimes \zeta_k + \text{Conjugate} , \\ (J e_r \lrcorner A)_g^T &= - A_{r,l,\bar{k}} \bar{\zeta}_l^* \otimes \zeta_k - \bar{A}_{k,l,\bar{r}} \zeta_l^* \otimes \zeta_k + \text{Conjugate} . \end{aligned}$$

(By conjugate we mean the complex conjugate of all therms preceding this word.) Thus hold the equality

$$\text{Tr}_{\mathbb{R}} \left[(e_r \lrcorner A) (e_r \lrcorner A)_g^T \right] = 2 |A_{l,k,\bar{r}}|^2 = \text{Tr}_{\mathbb{R}} \left[(J e_r \lrcorner A) (J e_r \lrcorner A)_g^T \right] .$$

We infer the identity

$$2 |A|_{\Lambda^2 T_X \otimes_{\mathbb{R}} T_X, g}^2 = |A|_{\Lambda_J^{1,1} T_X^* \otimes_{\mathbb{C}} T_{X,J}, \omega}^2 .$$

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